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Darboux Approach to M_α-integration

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ABSTRACT

It is known that one can develop Riemann integration theory via Darboux approach. The main idea in the Darboux approach is to define an integral using upper and lower Riemann sums. In this study we look at how M_α-integration can be develop via Darboux approach. Here is a brief discussion of the methodology. We define an equivalence relation on the set of M_α-divisions of $[a,b]$ such that for M_α- divisions $D_1 = \{([u,v],\xi)\}$ and $D_2 = \{([s,t],\eta)\}$ we say that $D_1 \sim D_2$ if and only if the intervals in D_1 are exactly the intervals in D_2. Given a gauge δ on $[a,b]$ and a δ-fine division $D = \{([u,v],\xi)\}$ of $[a,b]$, we set

$$[D,\delta] = \{P: P \sim D \text{ and } P\ is\ \delta - \text{fine } M_\alpha - \text{division}\}.$$

Given a function f on $[a,b]$, and a δ-fine M_α- division D, we define the upper and lower sums (respectively) in the following manner

$$S_\alpha^u(f,\delta,D) = \sup_{P\in[D,\delta]} (D)\sum f(\xi)(v-u) \text{ and } S_\alpha^l(f,\delta,D) = \inf_{P\in[D,\delta]} (D)\sum f(\xi)(v-u),$$

provided these values exists. We were able to show that a function f on $[a,b]$ is M_α-ntegrable if and only if the following exists and are equal:

$$(M_\alpha)\overline{\int_a^b} f = \inf_\delta \sup_D S_\alpha^u(f,\delta,D) \text{ and } (M_\alpha)\underline{\int_a^b} f = \sup_\delta \inf_D S_\alpha^l(f,\delta,D)$$

In this approach we were able to prove the basic properties of the M_α-integral. It is our next goal to extend M_α-integration to other spaces via Darboux approach.

Keywords: M_α-division, M_α-integral, Darboux approach.

INTRODUCTION

Recently, a new integral was introduced by Park, Ryu, and Lee in (Park et al., 2010), which they called M_α-integral. This integral, being equivalent to the C-integral discovered by Bongionio, provides further enlightenment to it. Recall that in (Bongiorno et al., 2010), C-integral is a minimal Henstock type constructive integration process, which can handle Lebesgue integrable functions and derivatives. The M_α-integral is also a Henstock type integral. Briefly, when we say Henstock-type integral, it has the aspect corresponding to the main component of the Henstock integral, the δ-fine divisions. In the case of M_α-integral, we have the δ-fine M_α-divisions. The Henstock integral is a generalization of the Riemann integral and that some authors would consider Henstock integral as a Riemann type integral. It considers a positive function $\delta(\cdot)$ called gauge instead of a positive number δ. Yet simple, this generalization has provided a brighter future for the analysis people. Meanwhile, as seen in standard introduction to real analysis textbooks, Riemann integration can also be developed via Darboux approach – the upper and lower integrals. This idea also holds for the Henstock integral, see for example the paper of Lee and Zhao (Lee and Zhao 1997). It is now our goal to develop a Darboux approach for the M_α-integration.

METHODS

The Darboux Approach to Riemann and Henstock Integrals

Given a closed and bounded interval $[a,b]$ a partition $D=\{[u_i,v_i]\}_{i=1}^{k}$ of $[a,b]$ is a finite collection of subinterval $[u_i,v_i]$ of $[a,b]$ whose union is $[a,b]$. That is

$$\bigcup_{i=1}^{k}[u_i,v_i]=[a,b].$$

In what follows, for convenience, instead of using $D=\{[u_i,v_i]\}_{i=1}^{k}$ to denote a partition, we shall be using $D=\{[u,v]\}$.

Recall that a function f on $[a,b]$ is said to be Riemann integrable with integral A if for every $\varepsilon>0$, there exists a positive number δ such that for any partition $D=\{[u,v]\}$ satisfying $\max\{v-u:[u,v]\in D\}<\delta$, we have

$$|(D)\sum f(\xi)(v-u)-A|<\varepsilon.$$

Where $\xi\in[u,v]$. Here, $(D)\sum f(\xi)(v-u)$ denotes the Riemann sum of f over D.

Let f be a bounded function on $[a,b]$ and $D=\{[u,v]\}$ be a partition of $[a,b]$. The upper Darboux sum of f over D is given by

$$S^{+}(f,D)=(D)\sum\sup_{\xi\in[u,v]}\{f(x)\}(v-u)$$

and the lower Darboux sum is

$$S_{-}(f,D)=(D)\sum\inf_{\xi\in[u,v]}\{f(x)\}(v-u).$$

A bounded function f on $[a,b]$ is said to be Darboux integrable if the following values exist and are equal $(D^*)\overline{\int_a^b f}=\inf_D S^{+}(f,D)$ and $(D^*)\underline{\int_a^b f}=\sup_D S_{-}(f,D)$.

The value A in which the two values coincide is the Darboux integral of f and we write

$$(D^*)\int_a^b f = A.$$

It is known that the Darboux integrability is equivalent to Riemann integrability and the two integrals coincide. See for example (Protter and Morrey 1991), for a more comprehensive discussion.

A partial division $D = \{([u,v],\xi)\}$ of $[a,b]$ is a finite collection of interval-point pairs $([u,v],\xi)$ where $\xi \in [u,v]$ and the subintervals $[u,v]$ of $[a,b]$ are non-overlapping which union is a subset of $[a,b]$. If in case the union of the subintervals $[u,v]$ is $[a,b]$ it self, then D is simply called a division of $[a,b]$. A gauge δ on $[a,b]$ is a positive function on $[a,b]$. A partial division $D = \{([u,v],\xi)\}$ is said to be δ-fine if for every $([u,v],\xi) \in D$, we have

$$[u,v] \subset \big(\xi - \delta(\xi), \xi + \delta(\xi)\big).$$

Whenever the containment above holds, the pair $([u,v],\xi)$ is δ-fine. we say Note that since δ is a positive function on $[a,b]$, the $\delta(\xi)$ above is a positive number. Given a gauge δ on $[a,b]$, the existence of a δ-fine division is guaranteed by the Cousin's Lemma. See (Lee 1989). We are now ready to define the Henstock integral. A function f on $[a,b]$ is said to be Henstock integrable with integral A if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine division $D = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$|(D)\sum f(\xi)(v-u) - A| < \varepsilon.$$

For a gauge δ and a δ-fine pair $([u,v],\xi)$, set $\delta_{([u,v],\xi)}$ as

$$\delta_{([u,v],\xi)} = \{x\colon ([u,v],\xi) \text{ is } \delta - \text{fine}\}.$$

Consider the following expressions

$$S^u(f,\delta,D) = (D)\sum\left(\sup_{x\in\delta_{(\xi,[u,v])}} f(x)\right)(v-u) \tag{1}$$

and

$$S_l(f,\delta,D) = (D)\sum\left(\inf_{x\in\delta_{(\xi,[u,v])}} f(x)\right)(v-u) \tag{2}$$

It follows from (Lee and Zhao 1997) that a function f on $[a,b]$ is Henstock integrable if and only if the following exists and are equal:

$(H)\overline{\int_a^b f} = \inf\limits_{\delta}\sup\limits_{D} S^u(f,\delta,D)$ and $(H)\underline{\int_a^b f} = \sup\limits_{\delta}\inf\limits_{D} S_l(f,\delta,D)$.

The values $(H)\overline{\int_a^b f}$ and $(H)\underline{\int_a^b f}$ are the upper and lower Henstock integral, respectively.

Let a positive number α be fixed. An M_α-partial division $D = \{([u,v],\xi)\}$ of $[a,b]$ is a finite collection of point interval pairs $([u,v],\xi)$ where $\xi \in [a,b]$, the subintervals $[u,v]$ of $[a,b]$ are non-overlapping and

$$(D)\sum \text{dist}\,(\xi,[u,v]) < \alpha. \tag{3}$$

If in case the union of the subintervals $[u,v]$ is $[a,b]$ then D is simply called an M_α- division. Recall that a function f on $[a,b]$ is said to be M_α-integrable if for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine M_α-division $D = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$|(D)\sum f(\xi)(v-u) - A| < \varepsilon.$$

While M_α-integration is very much similar to Henstock, the strategy applied to the Darboux approach to Henstock integration cannot be easily extended to M_α-integration. Observe that for a δ-fine division $D = \{([u,v],\xi)\}$ if one will modify D by picking one pair $(\xi,[u,v])$ and replacing ξ with $\xi^* \in [u,v]$ such that

$$[u,v] \subset \left(\xi^* - \delta(\xi^*), \xi^* + \delta(\xi^*)\right)$$

then the resulting division is still δ-fine and that (1) and (2) works. But for a particular δ-fine M_α-division $P = \{([u,v],\xi)\}$, if we are trying to modify P in similar manner, we may need to the check the rest of the pairs $([u,v],\xi)$ to maintain (3). This hinders the direct extension of the strategy applied to Henstock integration to M_α-integration. Hence a new approach is necessary.

Darboux Approach to M_α^*-IntegratioN

The key to our Darboux approach to M_α-integration, is to provide a structure on the set of M_α-divisions by defining an equivalence relation. For M_α-divisions $D_1 = \{([u,v],\xi)\}$ and $D_2 = \{([s,t],\eta)\}$ we say that $D_1 \sim D_2$ if and only if the intervals in D_1 are exactly the intervals in D_2. Given a gauge δ on $[a,b]$ and a δ-fine M_α-division $D = \{([u,v],\xi)\}$ of $[a,b]$, we set

$$[D,\delta] = \{P : P \sim D \text{ and } P\ is\ \delta - \text{fine } M_\alpha - \text{division } \}.$$

Given a function f on $[a,b]$, we define the M_α-upper and M_α-lower sums, respectively, in the following manner

$$S_\alpha^u(f,\delta,D) = \sup_{P\in[D,\delta]} (D)\sum f(\xi)(v-u) \text{ and } S_\alpha^l(f,\delta,D) = \inf_{P\in[D,\delta]} (D)\sum f(\xi)(v-u),$$

provided these values exists. We define the upper and lower M_α^*-integrals as

$$(M_\alpha^*)\overline{\int_a^b f} = \inf_\delta \sup_D S_\alpha^u(f,\delta,D) \text{ and } (M_\alpha^*)\underline{\int_a^b f} = \sup_\delta \inf_D S_\alpha^l(f,\delta,D),$$

repectively.

Definition 1 A function f on $[a,b]$ is said to be M_α^*-integrable integrable if the upper and lower M_α^*-integrals exist and are equal. Here the M_α^*-integral of f on $[a,b]$, denoted by $(M_\alpha^*)\int_a^b f$ is given by

$$(M_\alpha^*)\int_a^b f = (M_\alpha^*)\overline{\int_a^b} f = (M_\alpha^*)\underline{\int_a^b} f.$$

The Definition 1 presents our Darboux approach to M_α-integration. Before we look at the main agenda of this paper, which is proving its equivalence to M_α-integral, we will first look at some important properties of an integral that also holds for M_α^*-integral. Let us start with the Cauhy criterion.

Theorem 1 (Cauchy Criterion) A function f on $[a,b]$ is M_α^*-integrable if and only if given $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine M_α-division $D_1 = \{([u,v],\xi)\}$ and $D_2 = \{([s,t],x)\}$, we have

$$|S_\alpha^u(f,\delta,D_1) - S_\alpha^l(f,\delta,D_2)| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since $(M_\alpha^*)\overline{\int_a^b} f = \inf_\delta \sup_D S_\alpha^u(f,\delta,D)$, there exists a gauge δ_1 such that for any δ-fine M_α-division D of $[a,b]$, we have

$$\left| S_\alpha^u(f,\delta_1,D) - (M_\alpha^*)\overline{\int_a^b} f \right| < \frac{\varepsilon}{2} \tag{4}$$

and correspondingly since $(M_\alpha^*)\underline{\int_a^b} f = \sup_\delta \inf_D S_\alpha^l(f,\delta,D)$, there exists a gauge δ_2 such that for any δ-fine M_α-division P of $[a,b]$, we have

$$\left| (M_\alpha^*)\overline{\int_a^b} f - S_\alpha^l(f,\delta_2,P) \right| < \frac{\varepsilon}{2}. \tag{5}$$

Note that by definition of M_α^*-integrability, $(M_\alpha^*)\overline{\int_a^b} f = (M_\alpha^*)\overline{\int_a^b} f$. Let δ be a gauge on $[a,b]$ such that

$$\delta(x) = \min\{\delta_1(x), \delta_2(x)\}.$$

Considering the manner δ is define, it follows that every δ-fine M_α-division is also δ_1-fine and δ_2 -fine. Hence for any δ -fine M_α -division $D_1 = \{([u,v],\xi)\}$ and $D_2 = \{([s,t],x)\}$, considering (4) and (5), we have

$$|S_\alpha^u(f,\delta,D_1) - S_\alpha^l(f,\delta,D_2)| = \left| S_\alpha^u(f,\delta_1,D) - (M_\alpha^*)\overline{\int_a^b} f \right|$$

$$+\left|(M_\alpha^*)\overline{\int_a^b} f - S_\alpha^l(f,\delta_2,P)\right|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

■

The usefulness of Theorem 1 is not just limited to the proofs of the basic properties; it is also in fact taking a very important role in the proof of the main result of this paper.

Lemma 2 Let $[u,v]$ be a subinterval of $[a,b]$, δ be a gauge on $[a,b]$, and $D_{[u,v]}$ be a δ-fine division of $[u,v]$. Then there exists a δ-fine division D of $[a,b]$ such that $D_{[u,v]} \subset D$.

The proof of Lemma 2 is straightforward so it is omitted.

Theorem 3 If a function f on $[a,b]$ is M_α^*-integrable on $[a,b]$ then it is M_α^*-integrable on any subinterval $[s,t]$ of $[a,b]$.

Proof. Let be $[s,t]$ a subinterval of $[a,b]$ and $\varepsilon > 0$. Since f is M_α^*-integrable on $[a,b]$ then by Theorem 1 there exists a gauge δ on $[a,b]$ such that for any δ-fine M_α-division $D_1 = \{([u,v],\xi)\}$ and $D_2 = \{([s,t],x)\}$, we have

$$|S_\alpha^u(f,\delta,D_1) - S_\alpha^l(f,\delta,D_2)| < \varepsilon. \tag{6}$$

We will be utilizing this δ and apply Theorem 1 to show the M_α^*-integrablity of f on $[s,t]$. Now, let $P_1 = \{([u,v],\xi)\}$ and $P_2 = \{([s,t],x)\}$ be δ-fine M_α-divisions of $[s,t]$. By Lemma 2, we may extend P_1 to a δ-fine M_α-division, say P_1^*. Similarly, P_2 to P_2^*. It follows from (6) that

$$|S_\alpha^u(f,\delta,P_1) - S_\alpha^l(f,\delta,P_2)| = |S_\alpha^u(f,\delta,P_1^*) - S_\alpha^l(f,\delta,P_2^*)| < \varepsilon.$$

■

The following two theorems can be proved in standard manner, so the proofs are omitted.

Theorem 4 If a function f on $[a,b]$ is M_α^*-integrable on $[a,c]$ and $[c,b]$ with $c \in (a,b)$ then it is M_α^*-integrable on $[a,b]$ with

$$(M_\alpha^*)\int_a^b f = (M_\alpha^*)\int_a^c f + (M_\alpha^*)\int_c^b f.$$

Theorem 5 If a function f on $[a,b]$ is M_α^*-integrable on $[a,b]$ and k is constant then kf is M_α^*-integrable on $[a,b]$ with

$$(M_\alpha^*)\int_a^b kf = k \cdot (M_\alpha^*)\int_a^b f.$$

RESULTS AND CONCLUSION

This section is intended to show that a function f on $[a,b]$ is M_α-ntegrable if and only if the following exists and are equal:

$(M_\alpha^*)\overline{\int_a^b f} = \inf_\delta \sup_D S_\alpha^u(f,\delta,D)$ and $(M_\alpha^*)\underline{\int_a^b f} = \sup_\delta \inf_D S_\alpha^l(f,\delta,D)$.

Equivalently, we have the following theorem.

Theorem 6 A function f on $[a,b]$ is M_α^*-integrable if and only if it is M_α-integrable.

Proof. Suppose that f is M_α-integrable. Then for for every $\varepsilon > 0$ there exists a gauge δ on $[a,b]$ such that for any δ-fine M_α-division $D = \{([u,v],\xi)\}$ of $[a,b]$, we have

$$|(D)\sum f(\xi)(v-u) - A| < \frac{\varepsilon}{2}. \tag{7}$$

Now, let $D_1 = \{([s,t],\xi)\}$ and $D_2 = \{([s,t],x)\}$ be δ-fine M_α-divisions. Since both $[D_1,\delta]$ and $[D_2,\delta]$ are collections of δ-fine M_α-divisions, considering (7), it follows that for any $P_1 = \{([u,v],\xi)\} \in [D_1,\delta]$ and any $P_2 = \{([s,t],x)\} = [D_2,\delta]$, we have

$$\begin{aligned} |(P_1)\sum f(\xi)(v-u) - (P_2)\sum f(x)(t-s)| &\le |(P_1)\sum f(\xi)(v-u) - A| \\ &\quad + |A - (P_2)\sum f(x)(t-s)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It follows that $|S_\alpha^u(f,\delta,D_1) - S_\alpha^l(f,\delta,D_2)| \le \varepsilon$. Making this inequality strictly less than, can be easily handled. It now follows from the Cauchy criterion that f is M_α^*-integrable. We will now proceed to the converse.

Suppose that f is M_α^*-integrable and $\varepsilon > 0$. By the Cauchy criterion, there exists a gauge δ on $[a,b]$ such that for any δ-fine M_α-division $D_1 = \{([u,v],\xi)\}$ and $D_2 = \{([s,t],x)\}$, we have

$$|S_\alpha^u(f,\delta,D_1) - S_\alpha^l(f,\delta,D_2)| < \varepsilon.$$

For each n, set δ_n to be a gauge on $[a,b]$ such that for any δ_n-fine M_α-division $D_1 = \{([u,v],\xi)\}$ and $D_2 = \{([s,t],x)\}$, we have

$$|S_\alpha^u(f,\delta,D_1) - S_\alpha^l(f,\delta,D_2)| < \frac{1}{n} \tag{8}$$

with $\delta_{n+1}(x) \le \delta_n(x)$ for all $x \in [a,b]$. Now, for each n, fix a δ_n-fine M_α-division, say $D^n = \{([u,v],\xi)\}$. In this case, for $m > n$, $D^m = \{([u,v],\xi)\}$ is also δ_n-fine M_α-division. It follows that for $m > n$,

$$|S_\alpha^u(f,\delta,D^m) - S_\alpha^l(f,\delta,D^n)| < \frac{1}{n} \tag{9}$$

and hence, considering (8) and (9), we have

$$
\begin{aligned}
|S_\alpha^u(f,\delta,D^m) - S_\alpha^u(f,\delta,D^n)| &\le |S_\alpha^u(f,\delta,D^m) - S_\alpha^l(f,\delta,D^n)| \\
&\quad + |S_\alpha^l(f,\delta,D^n) - S_\alpha^u(f,\delta,D^n)| \\
&< \frac{1}{n} + \frac{1}{n} = \frac{2}{n}.
\end{aligned}
$$

It follows that the sequence $\{S_\alpha^u(f,\delta,D^m)\}$ is a Cauchy sequence and therefore convergent. Let A^u be its limit. Using similar argument, $\{S_\alpha^l(f,\delta,D^m)\}$ can be shown to be convergent. Let A^l be its limit. Since for $m > n$, (9) holds, taking the limit as $m \to \infty$, we get

$$|A^u - S_\alpha^l(f,\delta,D^n)| \le \frac{1}{n}. \tag{10}$$

Correspondingly, if we then take the limit of the inequality above as $n \to \infty$, we get

$$A^u = A^l.$$

Set $A = A^u = A^l$. We will show that $(M_\alpha)\int_a^b f = A$. Given $\varepsilon > 0$, let n be such that $\frac{3}{n} < \varepsilon$. Consider δ_n. Then for any δ_n-fine M_α-division D, we have

$$|S_\alpha^u(f,\delta,D) - (D)\sum f(\xi)(v-u)| \le |S_\alpha^u(f,\delta,D) - S_\alpha^l(f,\delta,D)|.$$

It now follows from (8) and (10) that

$$
\begin{aligned}
|A - (D)\sum f(\xi)(v-u)| &\le |A^u - S_\alpha^l(f,\delta,D^n)| + |S_\alpha^l(f,\delta,D^n) - S_\alpha^u(f,\delta,D)| \\
&\quad + |S_\alpha^u(f,\delta,D) - (D)\sum f(\xi)(v-u)| \\
&< \frac{1}{n} + \frac{1}{n} + \frac{1}{n} = \frac{3}{n} < \varepsilon.
\end{aligned}
$$

The Darboux approach presented in this paper is not real line dependent so it can be easily extended to higher dimension and the division space.

REFERENCES

Park, J. M., H. W. Ryu and H. K. Lee. (2010). The M_α-Integral, *J. Chungcheong Math. Soc. 23*(1), 99-108.

Lee, P.Y., and D. Zhao. (1997). Upper and Lower Henstock Integrals, *Real Analysis Exchange, 22*(2), 734-739.

Protter, M.H., and C.B. Morrey. (1991). *A First Course in Real Analysis (2nd ed.)*. New York: Springer-Verlag New York, Inc..

Lee, P. Y., (1989). *Lanzhou Lectures in Henstock Integration*. Singapore: World Scientific.

Bongiorno, B., L. Di Piazza, and D. Preiss. (2000). A constructive minimal integral which includes Lebesgue integrable functions and derivatives, *J. Lond. Math. Soc., II. Ser. 62*, 117–126.