Primitives of Essentially Bounded Henstock-Kurzweil Integrable Functions

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ABSTRACT

A full descriptive characterization of essentially bounded Henstock-Kurzweil integrable function is given. More precisely, an essentially bounded function f on $[a, b]$ is Henstock-Kurzweil integrable if and only if there exists a function F satisfying the Lipschitz condition on $[a,b]$ with $F'(x) = f(x)$ almost everywhere. Some implications were given, including integration by parts, substitution formula and a convergence theorem. These known results were presented and proved using the existing results in the Henstock-Kurzweil integration.

Keywords: Henstock-Kurzweil integral, essentially bounded functions.

INTRODUCTION AND LITERATURE REVIEW

A sequential definition for the Henstock-Kurzweil (*HK*) integral is possible as pointed out in (Lee, P. Y., 2007). A real valued function f on a compact interval $[a,b]$ in R is *HK*-integrable to a real number A, written $(HK)\int_a^b f = A$, if there is a sequence of positive functions $\delta_j, j = 1,2,3, \ldots$, such that for every δ_j-fine division $D_j = \{([u, v], \xi)\}$, we have

$$(D_j)\sum f(\xi)(v-u) \to A \; as \; j \to \infty.$$

An *HK*-integrable function f on $[a,b]$ is also (*HK*)-integrable to any subinterval of $[a,b]$. Hence we can define a function F such that $F(a) = 0$ and $F(x) = (HK)\int_a^x f$ for $a \in (a,b]$. This function is called the primitive of F. In Theorem 5.7 of (Lee, P. Y., 1989) and in Theorem 5.9 of (Bartle, R. G., 2001), it was shown that if F is primitive of f in $[a,b]$ then $F'(x) = f(x)$ almost everywhere on $[a, b]$.

There are several full descriptive characterization for the *HK*-integral. Among them is that a function f on $[a,b]$ is *HK*-integrable if there exists a function F satisfying the strong Lusin condition with $F'(x) = f(x)$ almost everywhere. For the discussion please see (Lee, P. Y. & Vyborny, R., 2000). It was then shown in (Bongiorno, B & Piazza, L., 1996) that a function satisfying the strong Lusin condition is differentiable almost everywhere. Hence a function F is the primitive of an *HK*-integrable function if and only if F satisfies the strong Lusin condition.

For easy reference, we state the following results.

Theorem 1.1. *Let a function f on $[a,b]$ be HK-integrable with primitive F. Then $F'(x) = f(x)$ almost everywhere.*

Theorem 1.2. *A function f on $[a,b]$ is HK-integrable if and only if there exists a function F satisfying the the strong Lusin condition with $F'(x) = f(x)$ almost everywhere. In this case $(HK)\int_a^b f = F(b) - F(a)$.*

In (Calunod, I. D., and Garces, I. J. L., 2021) and (Thompson, B. S., 2014), independently, a characterization of the primitives of Riemann integrable functions was given. It was shown that a function F on $[a,b]$ is the primitive of a Riemann integrable function on $[a,b]$ if and only if F satisfies the Lipschitz condition and is strongly differentiable almost everywhere on $[a,b]$. We say that a function F on $[a,b]$ satisfies the Lipschitz condition if there exists a nonnegative number M such that for any subinterval $[u,v]$ of $[a,b]$ we have $|F(v) - F(u)| \leq M(v-u)$. A function satisfying the Lipschitz condition also satisfies the strong Lusin condition. Then the following statement holds.

Theorem 1.3. *If F satisfies the Lipschitz condition on $[a,b]$ then F is differentiable almost everywhere.*

Recently in (Indrati, Ch. R. and Aryati, L., 2016), a new integral, the Countably Lipschitz or *CL*-integral, was defined. A function *f* is *CL*-integrable on $[a,b]$ if there exists a function F satisfying the countably Lipschitz condition on $[a,b]$ ($F \in CLC([a,b])$) such that $F'(x) = f(x)$ almost everywhere. A function $F \in CLC([a,b])$ if there exists a countable collection $\{X_i\}$ of subsets of $[a,b]$ and a countable collection $\{M_i\}$ of nonnegative numbers such that for any i, for any $u,v \in X_i$, we have $|F(v) - F(u)| \leq M(v-u)$. It was shown that every *CL*-integrable function is also *HK*-integrable.

RESULTS AND DISCUSSION

We start this section by stating the definition of the main subject, that is, the set of essentially bounded functions. Considering the proof that Riemann primitives satisfy the Lipschitz condition suggests intuitively that the primitives of other bounded integrable functions would also satisfy the Lipschitz condition. Recall that a real valued function *f* on $[a,b]$ is said to be essentially bounded if there exists a bounded function g such that $f = g$ almost everywhere on $[a,b]$. We will denote $B([a,b])$ as the set of all essentially bounded function on $[a,b]$. The following result immediately follows from the definition of $B([a,b])$.

Proposition 2.1. *A real valued function $f \in B([a,b])$ if and only if there exists a subset Φ_f of $[a,b]$ with full measure such that f is bounded on Φ_f.*

For any subset $S \subset [a,b]$ and for a function *f* which is bounded on *S*, we set $m_f(S) = inf\{f(x) : x \in S\}$ and $M_f(S) = sup\{f(x) : x \in S\}$.

Theorem 2.2. *Let $f \in B([a,b])$. If f is HK-integrable with primitive F then for any subinterval $[u,v]$ of $[a,b]$, we have*

$$m_f([u,v] \cap \Phi_f) \leq \frac{F(v) - F(u)}{v-u} \leq M_f([u,v] \cap \Phi_f).$$

Proof. Since F is the HK-primitive of f and $[u,v] \subset [a,b]$ then there is a sequence of positive functions $\delta_j, j = 1,2,3,...$ such that for every δ_j-fine division $D_j = \{([u,v],\xi)\}$, we have

$$F(v) - F(u) = \lim_{j\to\infty} (D_j) \sum f(\xi)(v-u).$$

Therefore, since HK-integral is invariant under changing of values for measure zero sets, we have

$$m_f([u,v] \cap \Phi_f)(v-u) \le F(v) - F(u) \le M_f([u,v] \cap \Phi_f)(v-u).$$

Corollary 2.3. *Let $f \in B([a,b])$. If f is HK-integrable with primitive F, then F satisfies the Lipschitz condition.*

Proof. If $f \in B([a,b])$ and is HK-integrable with primitive F then there exists a HK-integrable bounded function g having F also as primitive. Then for any subinterval $[u,v]$ of $[a,b]$, we have

$$m_g([a,b]) \le m_g([u,v]) \le \frac{F(v) - F(u)}{v-u} \le M_g([u,v]) \le M_{g([a,b])}.$$

If M is the higher number between $|m_g([a,b])|$ and $|M_g([a,b])|$, then for any subinterval $[u,v]$ of $[a,b]$, we have $|F(v) - F(u)| \le M(v-u)$.

□

Corollary 2.4. *If F is a differentiable function on $[a,b]$ with bounded derivative, then F is Lipschitz.*

Corollary 2.5. *Let $f \in B([a,b])$. If f is HK-integrable with primitive F, then $F'(x)$ exists and is equal to $f(x)$ whenever f is continuous at x.*

Proof. Let $f \in B([a,b])$ be HK-integrable with primitive F and $x_0 \in [a,b]$ such that f is continuous at x_0. Then $\lim_{x\to x_0^+} f(x) = f(x_0)$ and that for x close enough to x_0, we have

$$m_f([x_0,x]) \le \frac{F(x) - F(x_0)}{x - x_0} \le M_f([x_0,x]).$$

Taking the limit as $x \to x_0$, we get

$$f(x_0) \le \lim_{x\to x_0+} \frac{F(x) - F(x_0)}{x - x_0} \le f(x_0).$$

Using similar argument, one can show that

$$f(x_0) \leq \lim_{x \to x_0 -} \frac{F(x) - F(x_0)}{x - x_0} \leq f(x_0).$$

Therefore $F'(x_0) = f(x_0)$.

Theorem 2.6. *If F satisfies the Lipschitz condition then there exists a bounded Henstock integrable function f with F as primitive.*

Proof. If F satisfies the Lipschitz condition then $F'(x)$ exists almost everywhere on $[a,b]$ and for some $M \geq 0$, $F'(x) \leq M$ whenever $F'(x)$ exists. Define a function f on $[a,b]$ such that $f(x) = F'(x)$ when $F'(x)$ exists and $f(x) = 0$, otherwise. Then f is a bounded HK-integrable with F as primitive.

Theorem 2.7. *Let f be HK-integrable function on* $[a,b]$ *with primitive F. Then* $f \in B([a,b])$ *if and only if F satisfies the Lipschitz condition.*

Proof. Let f be *HK*-integrable with primitive F. If $f \in B([a,b])$ then by Corollary 2.3, F satisfies the Lipschitz Condition.

For the converse, let f be *HK*-integrable with primitive F satisfying the Lipschitz condition. Then F is differentiable almost everywhere. Let $X \subset [a,b]$ such that $F'(x) = f(x)$. Also by F being Lipschitz, there exists a nonnegrative number M such that for any subinterval $[u,v]$ of $[a,b]$, $|F(v) - F(u)| \leq M(v - u)$. Hence for any $x \in X, f(x) = F'(x) \leq M$. Therefore f is essentially bounded.

□

Lemma 2.8. *If a function F on* $[a,b]$ *satisfies the Lipschitz condition then it satisfies the strong Lusin condition.*

Theorem 2.9. *Let* $f \in B([a,b])$. *Then f is HK-integrable if and only if there exists a function F satisfying the Lipschitz condition such that* $F'(x) = f(x)$ *almost everywhere. In this case,* $(HK) \int_a^b f = F(b) - F(a)$.

Proof. Let $f \in B([a,b])$. If f is *HK*-integrable then by Theorem 2.3, F satisfies the Lipschitz condition. Furthermore, by Theorem 1.1, $F'(x) = f(x)$ almost everywhere.

For the converse, suppose there exists a function F satissfying the Lipschitz condition such that $F'(x) = f(x)$ almost everywhere. By Lemma 2.8, F satisfies the strong Lusin condition.
Therefore, by Theorem 1.2, f is *HK*-integrable and $(HK) \int_a^b f = F(b) - F(a)$.

We also have the following as auxiliary results.

Theorem 3.1. (A verion of Cauchy extension). *Let f be a function on* $[a,b]$ *such that for all c* $\in (a,b)$, $f \in B([a,c])$ *and is HK-integrable on* $[a,c]$. *If*

$$-\infty < \lim_{c \to b} (HK) \int_a^b f = A < +\infty$$

then f is HK-integrable with $(HK) \int_a^b f = A$.

Proof. Let $F(a) = 0, F(x) = (HK) \int_a^x f$ when $x \in (a, b)$ and $F(b) = A$. Set an increasing sequence $\{c_n\}$ in $[a,b]$ such that $c_n \to b$ as $n \to \infty$. By *HK*-integrability of f on $[a,c_n]$ with $F(c_n) = (HK) \int_a^{c_n} f$, it follows from Theorem 1.1 that for each n, $F'(x) = f(x)$ almost everywhere in $[a,c_n]$. Furthermore, by Corollary 2.3, for each n, F satisfies the Lipschitz condition in $[a,c_n]$. With F being Lipschitz on each $[a,c_n]$ and on $\{b\}$, then F satisfies the countably Lipschitz condition in $[a,b]$. Therefore F satisfies the countably Lipschitz condition and $F'(x) = f(x)$ almost everywhere in $[a,b]$. Therefore f is *HK*-integrable. Furthermore,

$$(HK) \int_a^b f = F(b) - F(a) = \lim_{c \to b} (HK) \int_a^b f.$$

Theorem 3.2. (Integration by Parts). *If F and G satisfy the Lipschitz condition in* $[a,b]$ *and* $F'(x) = f(x), G'(x) = g(x)$ *almost everywhere then*

$$(HK) \int_a^b (Fg + fG) = F(b)G(b) - F(a)G(a).$$

Proof. Note that both F and G are continuous in $[a,b]$, hence bounded. Let K be a common bound for F and G. Then for any $[u,v] \subset [a,b]$, we have

$$\begin{aligned} |F(v)G(v) - F(u)G(u)| &\leq |F(v)G(v) - F(v)G(u)| + |F(v)G(u) - F(u)G(u)| \\ &\leq K(|G(v) - G(u)| + |F(v) - F(u)|) \end{aligned}$$

It follows that FG satisfies the Lipschitz condition in $[a,b]$. Since $(FG)'(x) = F(x)g(x) + f(x)G(x)$ almost everywhere and product of two bounded functions is a bounded function, the result follows from Theorem 2.9.

Theorem 3.3. *Let* $g \in B([a,b])$ *be HK-integrable with primitive G. If* $f: G([a,b]) \to \mathbb{R}$ *is in* $B([a,b])$ *and is HK-integrable then* $(f \circ G)g$ *is integrable and*

$$\int_{G(a)}^{G(b)} f = \int_a^b (f \circ G) g.$$

Proof. Note that G is Lipschitz and $G'(x) = g(x)$ almost everywhere on $[a,b]$. Let $M_G \geq 0$ such that for any $[u,v] \subset [a,b]$, we have $|G(v) - G(u)| \leq M_G |v - u|$. Also, since f is HK-integrable on $G([a,b])$, its primitive, say F, is Lipschitz and differentiable almost everywhere on $G([a,b])$ to f. Let $M_F \geq 0$ such that for any $[u,v] \subset G([a,b])$, we have $|F(v) - F(u)| \leq M_F |v - u|$. Hence

$$|(F \circ G)(v) - (F \circ G)(u)| \leq M_F M_G |v - u|$$

Therefore $F \circ G$ satisfy the Lipschitz condition, and hence differentiable almost everywhere, and in fact, differentiable almost everywhere to $(f \circ G)g$. By Theorem 2.9, the result follows.

A sequence (F_n) of functions on $[a,b]$ is said to be equi-Lipschitz if there exists $M \geq 0$ such that for any $[u,v] \subset [a,b]$, for any n, we have $|Fn(v) - Fn(u)| \leq M|(v - u)$.

Theorem 3.4. *Let (F_n) be a sequence of functions on $[a,b]$ which is equi-Lipschitz. Then there exists a function $f \in B([a,b])$ that is HK-integrable and a subsequence of (F_i) of (F_n) such that such that*

$$\lim_{i\to\infty} F_i(b) - F_i(a) = (HK)\int_a^b f\,.$$

Proof. Since (F_n) is equi-Lipschitz, it is equi-continuous and therefore uniformly bounded. By Arzela-Ascoli's theorem it has a uniformly convergent subsequence. For convenience in notation, let us just denote the uniformly convergent subsequence by (F_n). Let F be the limit of (F_n). Then F is Lipschitz and therefore differentiable almost everywhere. Define a function f on $[a,b]$ such that $f(x) = F'(x)$ when $F'(x)$ exits and $f(x) = 0$ otherwise. Then f is boounded and HK-integrable with primitive F. In particular $\lim_{i\to\infty} F_i(b) - F_i(a) = (HK)\int_a^b f$.

CONCLUSION, IMPLICATION, SUGGESTION, AND LIMITATIONS

The paper suggests that a full discussion of integrability of essentially bounded functions via Henstock-Kurzweil theory of integration is possible. One may explore convergence theorems for this set of integrable functions in the future.

REFERENCES

Bartle, R. G. (2001). A modern theory of integration. Graduate Studies in Mathematics, 32. American Mathematical Society.

Bongiorno, B., & Piazza, L. (1996). A new full descriptive characterization of the Denjoy-Perron integral. Real Analysis Exchange, 21(2), 656-665.

Calunod, I. D., & Garces, I. J. L. (n.d.). Strong derivative and the essentially Riemann integral. Real Analysis Exchange. (To appear).

Indrati, Ch. R., & Aryati, L. (2016). The countably Lipschitz integral. Global Journal of Pure and Applied Mathematics, 12(5), 3991-3999.

Lee, P. Y. (1989). Lanzhou lectures on Henstock integration. World Scientific Publishing.

Lee, P. Y. (2007). The integral à la Henstock. Scientiae Mathematicae Japonicae Online, e-2007, 763-771.

Lee, P. Y., & Vyborny, R. (2000). The integral: An easy approach after Kurzweil and Henstock. Cambridge University Press.

Thompson, B. S. (2014). Strong derivatives and integrals. Real Analysis Exchange, 39(2), 469-488.