Theorems On n^{th} Dimensional Laplace Transform

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Abstract.

Let **U** be the set of all functions from $[0,\infty)^n$ to R and **V** be the set of all functions from $S \subseteq C^n$ to C.Then the n^{th} dimensional Laplace transform is the mapping

$$L^n : \mathbf{U} \to \mathbf{V}$$

defined by:

$$f(\tilde{\mathbf{s}}_n) = L\{\mathbf{F}(\tilde{\mathbf{x}}_n), \tilde{\mathbf{s}}_n\} = \int_{\mathbf{R}^n} \mathbf{F}(\tilde{\mathbf{x}}_n) e^{-(\tilde{\mathbf{s}}_n \cdot \tilde{\mathbf{x}}_n)} d\tilde{\mathbf{x}}_n$$

Where $\mathbf{F}(\tilde{\mathbf{x}}_n) \in \mathbf{U}$ and $\tilde{\mathbf{s}}_n \in C^n$

In this paper we gave alternative proof for some theorems on properties of n^{th} dimensional Laplace Transform, we proved that if $\mathbf{F}(\tilde{\mathbf{x}}_n)$ is piecewise continuous on $[0,\infty)^n$ and function of exponential order $\tilde{\gamma}_n = (\gamma_1, \gamma_2, ..., \gamma_n)$ then the n^{th} dimensional Laplace Transform defined above exists, absolutely and uniformly convergent, analytic and infinitely differentiable on $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2, ..., Re(s) > \gamma_n$, and we gave also some corollaries of these results.

I. Introduction

Laplace Transform is a widely used integral transform with many applications in Physics, Engineering and Applied Mathematics. It is named after Pierre-Simon Laplace, who introduced the transform in his work on probability theory. It is also very useful for solving ordinary and partial differential equations, difference equations, integral equations as well as mixed equations. In physics and engineering it is used for analysis of linear time-invariant systems such as electrical circuits, harmonic oscillators, optical devices, and mechanical sytems, It can be used as a very effective tool in simplifying the calculations in many fields of engineering and mathematics. It also provides a powerful method for analyzing linear systems.

Estrin and Higgins [21] in 1951, has given the systematic account of the general theory of an operational calculus using double Laplace transform and illustrated its application through solution of two problems, one in electrostatics, the other in heat conduction. Coon and Bernstein [22] conducted systematic study of double Laplace transform in 1953 and developed its properties including conditions for transforming derivatives, integrals and convolution. Buschman [23] in 1983 used double Laplace transform to solve a problem on heat transfer between a plate and a fluid flowing across the plate. In 1989, Debnath and Dahiya [11], [12] established a set of new theorems concerning multidimensional Laplace transform of some functions and used the technique developed to solve an electrostatic potential problem. In 1992, Yu A. Brychov, H. J Glaeske, A.P Prudnikov, Vu Kim Tuan [27] in their book entitled ”Multidimensional Integral Transformations” discussed and proved several theorems on the properties of n^{th} dimensional Laplace transform. Recently in 2013, Aghili and Zeinali [24] implemented multidimensional Laplace transforms method for solving certain non-homogeneous forth order partial differential equations, Abdon Atangana [26] discussed some properties of triple Laplace transform and applied to some kind third order partial differential equations. In 2014, Eltayeb, Kilicman and Mesloub [18] applied double Laplace transform method to evaluate the exact value of double infinite series. Hamood Ur Rehman, Muzammal Iftikhar, Shoaib Saleem, Muhammad Younis, Abdul Mueed [19] discussed some properties and theorems about the quadruple Laplace transform and gave a good strategy for solving the fourth order partial differential equations in engineering and physics fields, by quadruple Laplace transform, Ranjit R. Dhunde and G. L. Waghmare [20] presented the absolute convergence and uniform convergence of double Laplace Transform and solved a Volterra IntegroPartial Differential Equation using double Laplace transform.

The main objective of this paper is to extend some results in [10],[19],[20],[24] and [26], provide alternative proof for some results in [27] on theorems on the properties of the n^{th} dimesional Laplace transform and derive some special multiple improper integral identities.

1. The n^{th} Dimensional Laplace Transform

In this section we will deal with the main results. For brevity, we will use the following notation throughout this chapter. $\mathbf{F}(\tilde{\mathbf{x}}_n)=\mathbf{F}(x_1,x_2,...,x_n)$; $f(\tilde{\mathbf{s}}_n)=f(s_1,s_2,...,s_n)$; $\tilde{\mathbf{s}}_n=(s_1,s_2,...,s_n)$; $\tilde{\mathbf{x}}_n=(x_1,x_2,...,x_n)$; $\mathbf{d}\tilde{\mathbf{x}}_n=dx_1dx_2...dx_n$;

$$\tilde{\mathbf{s}}_n\cdot\tilde{\mathbf{x}}_n=\sum_{i=1}^{n}s_ix_i$$

$$\mathbf{R}_n=[0,\infty)\times[0,\infty)\times...\times[0,\infty)=[0,\infty).$$

Definition 1. *Let* $\mathbf{U}$ *be the set of all functions from* $[0,\infty)^n$ *to* $\mathbb{R}$ *and* $\mathbf{V}$ *be the set of all functions from* $S\subseteq\mathbb{C}^n$ *to* $\mathbb{C}$*. Then the* n^{th} *dimensional Laplace transform is the mapping*

$L^n:\mathbf{U}\to\mathbf{V}$

defined by:

$$L^n\{\mathbf{F}(\tilde{\mathbf{x}}_n),\tilde{\mathbf{s}}_n\}=\int_{\mathbf{R}}\mathbf{F}(\tilde{\mathbf{x}}_n)\mathbf{e}^{-(\tilde{\mathbf{s}}_n\cdot\tilde{\mathbf{x}}_n)}\mathbf{d}\tilde{\mathbf{x}}_n$$

Where $\mathbf{F}(\tilde{\mathbf{x}}_n)\in\mathbf{U}$ *and* $\tilde{\mathbf{s}}_n\in\mathbb{C}^n$.

From the above definition, it is clear that

$$\tilde{\mathbf{s}}_n\cdot\tilde{\mathbf{x}}_n=\sum_{i=1}^{n}s_ix_i=\sum_{i=1}^{n}a_ix_i+i\sum_{i=1}^{n}b_ix_i$$

if $s_i=a_i+ib_i$ *for* $i=1,2,3,...,n$.

Occasionally, we also denote

$f(\tilde{\mathbf{s}}_n)=L^n\{\mathbf{F}(\tilde{\mathbf{x}}_n),\tilde{\mathbf{s}}_n\}$

It is also clear the there are some functions in $\mathbf{U}$ in which the n^{th} dimensional Laplace transform does not exist. We now define a class of functions needed for the existence of the n^{th} dimensional Laplace transform.

Definition 2. *A function* $\mathbf{F}(\tilde{\mathbf{x}}_n)$ *is called piecewise continuous in the region* $S\subseteq\mathbb{R}^n$ *if the region can be divided into a finite number of subregion such that* $\mathbf{F}(\tilde{\mathbf{x}}_n)$ *is continuous.*

Definition 2 is a natural extension to n^{th} dimension of piecewise continuity and from this definition we can evaluate the n^{th} dimensional Laplace Transform using theorems on multiple integrals. We now Define other class of functions needed for the existence of n^{th} dimensional Laplace transform.

Definition 3. *If positive real constant* $\mathbf{M}$ *and a vector* $\tilde{\gamma}_n=(\gamma_1,\gamma_2,...,\gamma_n)$ *in* $\mathbb{R}^n$ *exist such that at least one i of* $x_i>N_i$ *is true for* $i=1,2,3...,n$ *we have:*

$|\mathbf{F}(\tilde{\mathbf{x}}_n)\mathbf{e}^{-(\tilde{\gamma}_n\cdot\tilde{\mathbf{x}}_n)}|<\mathbf{M}$ *or*

$|\mathbf{F}(\tilde{\mathbf{x}}_n)|<\mathbf{M}\mathbf{e}^{(\tilde{\gamma}_n\cdot\tilde{\mathbf{x}}_n)}$ *then* $\mathbf{F}(\tilde{\mathbf{x}}_n)$ *is said to be function of exponential order* $\tilde{\gamma}_n$.

Definition 3 is an extension to n^{th} dimension for the functions of exponential order. Functions of exponential order cannot grow in absolute value more rapidly than $\mathbf{M}\mathbf{e}^{(\tilde{\gamma}_n\cdot\tilde{\mathbf{x}}_n)}$ if at least one of $x_i'^s$ increases.

Clearly the exponential function $e^{a_1x_1+a_2x_2+...+a_nx_n}$ has exponential order $\gamma = (a_1,a_2,...,a_n)$, whereas $\prod_{k=1}^{m} x_k^{n_k}$ has exponential order $\gamma = (\gamma_1,\gamma_2,...,\gamma_n)$ where at least one of $\gamma_i > 0$ and any $n_k \in \mathbb{N}$ for all k.
Bounded functions like $\sin(\mathbf{P}(\tilde{\mathbf{x}}_\mathbf{n}))$, $\cos(\mathbf{Q}(\tilde{\mathbf{x}}_\mathbf{n}))$, $\tan^{-1}(\mathbf{P}(\tilde{\mathbf{x}}_\mathbf{n}))$ have exponential order $\gamma = (0,0,...,0)$ where $\mathbf{P}(\tilde{\mathbf{x}}_\mathbf{n}),\mathbf{Q}(\tilde{\mathbf{x}}_\mathbf{n})$ and $\mathbf{H}(\tilde{\mathbf{x}}_\mathbf{n})$ are continuous functions from $\mathbb{R}^n$ to R while $e^{-(x_1+x_2+...+x_n)}$ has order $\gamma = (-1,-1,...,-1)$. However,

2. Convergence Theorem

The following theorem gives us sufficient conditions for the existence of n^{th} dimensional Laplace transform.

Theorem 1. *If* $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ *is piecewise continuous in every bounded region and of exponential order* $\tilde{\gamma}_\mathbf{n}$ *then the* n^{th} *dimensional Laplace Transform* $f(\tilde{\mathbf{s}}_\mathbf{n})$ *exists for all* $Re(s_1) > \gamma_1$, $Re(s_2) > \gamma_2$,..., $Re(s_n) > \gamma_n$.

Proof:

From Definition 1 we have $f(\tilde{\mathbf{s}}_\mathbf{n}) =$
$L^n\{\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n}),\tilde{\mathbf{s}}_\mathbf{n}\} = \int_{\mathbf{R}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n}$

Hence we get: $\int_{\mathbf{R}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n} = \int_{\mathbf{R_1}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n} + \int_{\mathbf{R_2}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n}$, where $\mathbf{R_1}$ is a bounded region, $\mathbf{R_2}$ is an unbounded region such that $\mathbf{R} = \mathbf{R_1} \cup \mathbf{R_2}$ and
$\mathbf{R_1} \cap \mathbf{R_2} = \emptyset$.
Clearly the integral
$\int_{\mathbf{R_1}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n}$
exists, since $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ is piecewise continuous in every bounded region. so we only need to show that the integral $\int_{\mathbf{R_2}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n}$
exists. now we have a chain of inequality:

$$\left|\int_{\mathbf{R_2}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n}\right| \leq \int_{\mathbf{R_2}}\left|\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}\right|d\tilde{\mathbf{x}}_\mathbf{n} \leq \int_{\mathbf{R}}\left|\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}\right|d\tilde{\mathbf{x}}_\mathbf{n}$$
$$\leq \int_{\mathbf{R}}|\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})|\,|e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}|d\tilde{\mathbf{x}}_\mathbf{n}$$

but since $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ is of exponential order $\tilde{\gamma}_\mathbf{n}$,

$$\left|\int_{\mathbf{R_2}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n}\right| \leq \int_{\mathbf{R}}\mathbf{M}e^{(\tilde{\gamma}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}e^{-(Re(\tilde{\mathbf{s}}_\mathbf{n})\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n} \leq \int_{\mathbf{R}}\mathbf{M}e^{-[Re(\tilde{\mathbf{s}}_\mathbf{n})-\tilde{\gamma}_\mathbf{n}]\cdot\tilde{\mathbf{x}}_\mathbf{n}}d\tilde{\mathbf{x}}_\mathbf{n}$$

$\int_{\mathbf{R}}\mathbf{M}e^{-[Re(\tilde{\mathbf{s}}_\mathbf{n})-\tilde{\gamma}_\mathbf{n}]\cdot\tilde{\mathbf{x}}_\mathbf{n}}d\tilde{\mathbf{x}}_\mathbf{n}$ is equal to

$$\frac{\mathbf{M}}{\prod_{i=1}^{n}(Re(s_i)-\gamma_i)}.$$

Hence if $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2, ..., Re(s_n) > \gamma_n$ then:

$$\left|\int_{\mathbf{R_2}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n}\right| \leq \frac{\mathbf{M}}{\prod_{i=1}^{n}(Re(s_i)-\gamma_i)}$$

Therefore if $Re(s_i) > \gamma_i$, for $1 \leq i \leq n$, $f(\tilde{\mathbf{s}}_\mathbf{n})$ exists.
We now prove the absolute convergence of the n^{th} dimensional Laplace integral.

Theorem 2. *If* $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ *is piecewise continuous on* $[0,\infty)^n$ *and of exponential order* $\tilde{\gamma}_\mathbf{n}$ *then the* n^{th} *dimensional Laplace transform:*
$L^n\{\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n}),\tilde{\mathbf{s}}_\mathbf{n}\} = \int_{\mathbf{R}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}d\tilde{\mathbf{x}}_\mathbf{n}$
is absolutely convergent for $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2,...,Re(s_n) > \gamma_n$.

Proof:
From Theorem 1

$$\int_{\mathbf{R}}\left|\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})e^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}\right|d\tilde{\mathbf{x}}_\mathbf{n} \leq \int_{\mathbf{R}}\mathbf{M}e^{-[Re(\tilde{\mathbf{s}}_\mathbf{n})-\tilde{\gamma}_\mathbf{n}]\cdot\tilde{x}_n}d\tilde{\mathbf{x}}_\mathbf{n} \leq \frac{\mathbf{M}}{\prod_{i=1}^{n}(Re(s_i)-\gamma_i)}$$

if $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2,...,Re(s_n) > \gamma_n$. thus

$$L^n\{\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n}),\tilde{\mathbf{s}}_\mathbf{n}\} = \int_{\mathbb{R}} \mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n}) \mathbf{e}^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})} d\tilde{\mathbf{x}}_\mathbf{n}$$

is absolutely convergent for all $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2, ..., Re(s_n) > \gamma_n$.

Similar result for Theorem 2 can be found in [**22**] where the function $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ is considered a mapping from $[0,\infty)^n$ to C while in this result we considered our function $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ to be a mapping from $[0,\infty)^n$ to R with additional condition that the function $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ must be piecewise continuous on $[0,\infty)^n$. In this result we actually found the bound for the integral

$$\int_{\mathbf{R}} \left|\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})\,\mathbf{e}\right| d\tilde{\mathbf{x}}_\mathbf{n}.$$

We now prove two lemmas needed for the proof of the uniform convergence of the n^{th} dimensional Laplace transform.

Lemma 1. *If* $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ *is piecewise continuous on* $[0,\infty)^n$ *and of exponential order* $\tilde{\gamma}_\mathbf{n}$ *such that* $Re(s_i) > \gamma_i$, *then*

$$\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i) = \int_0^\infty e^{-s_i x_i}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})dx_i$$

is piecewise continuous on $[0,\infty)^{n-1}$ *and of exponential order* $\tilde{\gamma}_{\mathbf{n-1}}$, *Where* $\tilde{\mathbf{x}}_{\mathbf{n-1}}, \tilde{\gamma}_{\mathbf{n-1}}$ *are* $n-1$ *dimensional vectors without* x_i *and* γ_i.

Proof:

The function

$$\mathbf{H}(\tilde{\mathbf{x}}_\mathbf{n}, s_i) = e^{-s_i x_i}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$$

is piecewise continuous on $[0,\infty)^n$ since $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ is piecewise continuous on $[0,\infty)^n$, so it follows that $\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i)$ is also piecewise continuous on $[0,\infty)^{n-1}$. we need only to show that $\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i)$ is a function of exponential order $\tilde{\gamma}_{\mathbf{n-1}}$, now

$$|\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i)| = \left|\int_0^\infty e^{-s_i x_i}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})dx_i\right|$$

Hence

$$|\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i)| \le \int_0^\infty \left|e^{-s_i x_i}\right||\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})|dx_i$$

But is it clear that

$$\int_0^\infty \left|e^{-s_i x_i}\right||\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})|dx_i$$

is less than

$$\int_0^\infty e^{-x_i Re(s_i)}\mathbf{M}\,\mathbf{e}^{\sum_{k=1}^n \gamma_k x_k}\mathbf{e}^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}dx_i$$

Therefore

$|\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i)|$ is less than

$$\int_0^\infty e^{-x_i Re(s_i)}\mathbf{M}\,\mathbf{e}^{\sum_{k=1}^n \gamma_k x_k}\mathbf{e}^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}dx_i$$

for at least one i of $x_i > N_i$, $i = 1,2,3,...,n$ thus,

$$|\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i)| \le \mathbf{N}e^{\sum_{k\neq i}^n \gamma_k x_k}$$

That is, $\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i)$, is of exponential order $\tilde{\gamma}_{\mathbf{n-1}}$.

Lemma 2. *If* $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ *is piecewise continuous on* $[0,\infty)^n$ *and of exponential order* $\tilde{\gamma}_\mathbf{n}$ *then*

$$\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_i) = \int_0^\infty e^{-s_i x_i}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})dx_i$$

is uniformly convergent on $Re(s_i) > \gamma_i$ *for* $i = 1,2,...,n$.

Proof:

We know that

$$\left|\int_u^\infty e^{-s_i x_i}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})dx_i\right| \le \int_u^\infty \left|e^{-s_i x_i}\right||\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})|dx_i$$

and from Lemma 1 we have:

$$\left|\int_u^\infty e^{-s_i x_i}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})dx_i\right| \le \frac{\mathbf{M}}{Re(s_i)-\gamma_i}e^{-[Re(s_i)-\gamma_i]u}e^{\sum_{k\neq i}^n \gamma_k x_k}$$

provided $Re(s_i) > \gamma_i$ for $i = 1,2,...,n$.

whenever

$$\frac{\mathbf{M}}{Re(s_i)-\gamma_i}e^{-[Re(s_i)-\gamma_i]u}e^{\sum_{k\neq i}^n \gamma_k x_k} < \epsilon$$

then

$$u > \frac{\ln\left(\frac{\mathbf{M}e^{\sum_{k\neq i}^n \gamma_k x_k}}{(Re(s_i)-\gamma_i)\epsilon}\right)}{Re(s_i)-\gamma_i}$$

so we choose

$$N = \frac{\ln\left(\frac{\mathbf{M}e^{\sum_{k\neq i}^n \gamma_k x_k}}{(Re(s_i)-\gamma_i)\epsilon}\right)}{Re(s_i)-\gamma_i},$$

this N is always positive for $Re(s_i) > \gamma_i$, that is given any $\epsilon > 0$, there exists a value $N > 0$ such that

$$\left|\int_u^\infty e^{-s_i x_i}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})dx_i\right| < \epsilon$$

whenever $u > N$ for all values of s_i with $Re(s_i) > \gamma_i$. This is precisely the condition required for the uniform convergence in the region $Re(s_i) > \gamma_i$.

The following theorem proves the uniform convergence of the n^{th} Dimensional Laplace Transform.

Theorem 3. *If* $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ *is piecewise continuous on* $[0,\infty)^n$ *and of exponential order* $\tilde{\gamma}_\mathbf{n}$ *then*

$f(\tilde{\mathbf{s}}_\mathbf{n}) = L^n\{\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n}),\tilde{\mathbf{s}}_\mathbf{n}\}$

converges uniformly on $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2, ..., Re(s_n) > \gamma_n$.

Proof:

Without loss of generality, write $f(\tilde{\mathbf{s}}_\mathbf{n})$ as

$$\int_0^\infty ... \int_0^\infty e^{-\sum_{k=2}^n s_k x_k} G(s_1) \mathbf{d}\tilde{\mathbf{x}}_{\mathbf{n-1}}, \quad G(\vec{x}_{n-1}, s_1) = \left\{\int_0^\infty e^{-s_1 x_1}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})dx_1\right\}$$

where now by Lemma 2, $G(\tilde{x}_{n-1},s_1)$

$$\mathbf{G}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_1) = \int_0^\infty e^{-s_1 x_1}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})dx_1$$

converges uniformly on $Re(s_1) > \gamma_1$.. Hence $G(\tilde{x}_{n-1},s_1)$ is piecewise continuous, by Lemma 1, on $[0,\infty)^{n-1}$ and of exponential order $\tilde{\gamma}_{\mathbf{n-1}}$ so again by Lemma 2 the integral

$$\int_0^\infty e^{-s_2 x_2}\mathbf{H}(\tilde{\mathbf{x}}_{\mathbf{n-1}}, s_1)dx_2$$

converges uniformly on $Re(s_2) > \gamma_2$.

Thus, by applying Lemma 1 and Lemma 2 repeatedly we see that the integral

$$f(\tilde{\mathbf{s}}_\mathbf{n}) = \int_{\mathbf{R}} \mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})\, \mathbf{e}^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}\mathbf{d}\tilde{\mathbf{x}}_\mathbf{n}$$

is uniformly convergent on $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2, ..., Re(s_n) > \gamma_n$.

Similar result for Theorem 3 can be found in [**27**]. In our result we used two lemmas to prove the uniform convergence of the n^{th} dimensional Laplace integral.

In what follows, we will state that based on Theorem 3 the order of limit and integral can safely be reversed. The proof is straightforward.

Corollary 1. *If* $\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})$ *is piecewise continuous on* $[0,\infty)^n$ *and of exponential order* $\tilde{\gamma}_\mathbf{n}$ *then*

$$f(\tilde{\mathbf{s}}_\mathbf{n}) = \int_{\mathbf{R}} \mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})\, \mathbf{e}^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}\mathbf{d}\tilde{\mathbf{x}}_\mathbf{n}$$

is continuous and integrable complex valued function in real variables on $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2, ..., Re(s_n) > \gamma_n$.

Other implication of Theorem 3 is again stated as corollary.

Corollary 2. *If the integral* $\int_{\mathbf{R}}\mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})\mathbf{d}\tilde{\mathbf{x}}_\mathbf{n}$ *is convergent, then*

$$f(\tilde{\mathbf{s}}_\mathbf{n}) = \int_{\mathbf{R}} \mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})\, \mathbf{e}^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}\mathbf{d}\tilde{\mathbf{x}}_\mathbf{n}$$

converges uniformly on $Re(s_i) \geq 0$ *for* $i = 1,2,3,...,n$.

3. Behavior Of The n^{th} Dimensional Laplace Transform

From Corollary 1, we can insert the limit on any point on the region $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2, ..., Re(s_n) > \gamma_n$ inside the n^{th} dimensional Laplace integral and thus

$$\lim_{Re(\tilde{\mathbf{s}}_\mathbf{n})\to\infty} f(\tilde{\mathbf{s}}_\mathbf{n})$$

can be expressed as

$$\int_{\mathbf{R}} \lim_{Re(\tilde{\mathbf{s}}_\mathbf{n})\to\infty} \mathbf{F}(\tilde{\mathbf{x}}_\mathbf{n})\mathbf{e}^{-(\tilde{\mathbf{s}}_\mathbf{n}\cdot\tilde{\mathbf{x}}_\mathbf{n})}\mathbf{d}\tilde{\mathbf{x}}_\mathbf{n}$$

which can be further simplified as

$$\int (0)\mathbf{d}\tilde{\mathbf{x}}_\mathbf{n} = 0$$

4. Analyticity Of The n^{th} Dimensional Laplace Transform $f(\tilde{\mathbf{s}}_\mathbf{n})$

Analyticity plays an important role in complex analysis and in physics, since once the function was determined to be analytic in some region then its higher order derivatives are also analytic in the same region, also the real and imaginary parts of the function are infinitely many times differentiable and harmonic in the same region.

Before proving the analyticity of n^{th} dimensional Laplace transform $f(\tilde{\mathbf{s}}_\mathbf{n})$ we first prove the following lemmas.

Lemma 3. *If* $\mathbf{F}(\tilde{\mathbf{t}}_n)$ *is piecewise continuous on* $[0,\infty)^n$ *and of exponential order* $\tilde{\gamma}_n$ *then the*

$$a)\int_0^\infty e^{-s_it_i}t_i^{m_i}\cos\left(\sum_{k=1}^n y_kt_k\right)\mathbf{F}(\tilde{\mathbf{t}}_n)dt_i$$

n

integrals

!

$\mathbf{F}(\tilde{\mathbf{t}}_n)dt_i$

$$b)\int_0^\infty e^{-s_it_i}t_i^{m_i}\sin\left(\sum_{k=1} y_kt_k\right.$$

converges uniformly on (*i*) *i*, *where* $m_i \in Z^+$ *for* $i = 1,2,...,n$.

Proof of (a):

By direct evaluation we get

$$\left|e^{-s_it_i}t_i^{m_i}\cos\left(\sum_{k=1}^n y_kt_k\right)\mathbf{F}(\tilde{\mathbf{t}}_n)\right|$$

$$\leq \mathbf{M}e^{\sum_{k\neq i}^n \gamma_kt_k}t_i^{m_i}e^{-[Re(s_i)-\gamma_i]t_i}$$

for $t_i \geq 0$. But the integral

$$\int_0^\infty t_i^{m_i}e^{-[Re(s_i)-\gamma_i]t_i}dt_i$$

converges for $Re(s_i) > \gamma_i$. Thus by Weiertrass M test, the integral

$$\int_0^\infty e^{-s_it_i}t_i^{m_i}\cos\left(\sum_{k=1}^n y_kt_k\right)\mathbf{F}(\tilde{\mathbf{t}}_n)dt_i$$

converges uniformly on $Re(s_i) > \gamma_i$.

Proof of (b): The Proof is analogous to the proof of (a).

We now prove the analyticity of n^{th} dimensional Laplace transform $f(\tilde{\mathbf{s}}_n)$.

Theorem 4. *If* $\mathbf{F}(\tilde{\mathbf{t}}_n)$ *is piecewise continuous on* $[0,\infty)^n$ *and of exponential order* $\tilde{\gamma}_n$ *then*

$\tilde{Z}\ f(\tilde{\mathbf{s}}_n) =$ $\mathbf{F}(\tilde{\mathbf{t}}_n)\ e^{-(\tilde{\mathbf{s}}_n\cdot\tilde{\mathbf{t}}_n)}d\tilde{\mathbf{t}}_n$

$\mathbf{R}$ *is analytic function on* $Re(s_1) > \gamma_1,...,Re(s_n) > \gamma_n$.

Proof: Let $s_k = Re(s_k)+iRe(s_k) = x_k + iy_k$, then

$f(\tilde{\mathbf{s}}_n) = \mathbf{u}(\tilde{\mathbf{x}}_n,\tilde{\mathbf{y}}_n) + i\mathbf{v}(\tilde{\mathbf{x}}_n,\tilde{\mathbf{y}}_n)$.

where $\mathbf{u}(\tilde{\mathbf{x}}_n,\tilde{\mathbf{y}}_n)$ and $\mathbf{v}(\tilde{\mathbf{x}}_n,\tilde{\mathbf{y}}_n)$ are respectively expressed as

$$\tilde{\int_{\mathbf{R}}}\mathbf{F}(\tilde{\mathbf{t}}_n)\cos\left(\sum_{k=1}^n y_kt_k\right)e^{-\sum_{k=1}^n x_kt_k}\ d\tilde{\mathbf{t}}_n$$

and

n *k k* $d\tilde{\mathbf{t}}_n$

By Lemma 3 we can reverse the order of derivative and the integrals of the real and imaginary parts of $f(\tilde{\mathbf{s}}_n)$ and hence the integrals

$$Re\ s\quad > \gamma\qquad \tilde{\int_{\mathbf{R}}}\mathbf{F}(\tilde{t}\ )\sin\left(\sum_{k=1}^n y\ t\ \right)e^{-\sum_{k=1}^n x_kt_k}$$

$$\frac{\partial \mathbf{u}}{\partial x_j} = \tilde{\int_{\mathbf{R}}} - t_j\mathbf{F}(\tilde{\mathbf{t}}_n)R(\tilde{t}_n)d\tilde{\mathbf{t}}_n$$

$$\frac{\partial \mathbf{u}}{\partial y_j} = \tilde{\int_{\mathbf{R}}} - t_j\mathbf{F}(\tilde{\mathbf{t}}_n)S(\tilde{t}_n)d\tilde{\mathbf{t}}_n$$

$$\frac{\partial \mathbf{v}}{\partial x_j} = -\tilde{\int_{\mathbf{R}}} - t_j\mathbf{F}(\tilde{\mathbf{t}}_n)S(\tilde{t}_n)d\tilde{\mathbf{t}}_n$$

$$\frac{\partial \mathbf{v}}{\partial y_j} = -\tilde{\int_{\mathbf{R}}} t_j\mathbf{F}(\tilde{\mathbf{t}}_n)R(\tilde{t}_n)d\tilde{\mathbf{t}}_n$$

where

$R(\vec{t}_n) = \cos\left(\sum_{k=1}^n y_kt_k\right)e^{-\sum_{k=1}^n x_kt_k}$

and

$S(\vec{t}_n) = \sin\left(\sum_{k=1}^n y_kt_k\right)e^{-\sum_{k=1}^n x_kt_k}$,

are uniformly convergent on $x_k = Re(s_k) > \gamma_k$ for $k = 1,2,3,...,n$.

Thus

$$\frac{\partial \mathbf{u}}{\partial x_j},\ \frac{\partial \mathbf{u}}{\partial y_j},\ \frac{\partial \mathbf{v}}{\partial x_j},\ \frac{\partial \mathbf{v}}{\partial y_j}$$

are continuous on $x_k = Re(s_k) > \gamma_k$ for $k = 1,2,3,...,n$ and also, we see that:

$$\frac{\partial \mathbf{u}}{\partial x_j} = \frac{\partial \mathbf{v}}{\partial y_j}\quad and\quad \frac{\partial \mathbf{u}}{\partial y_j} = -\frac{\partial \mathbf{v}}{\partial x_j}$$

that is, Cauchy-Riemann equations are satiesfied on $x_k = Re(s_k) > \gamma_i$ for $k = 1,2,3,...,n$.

Thus

$\tilde{Z}\ f(\tilde{\mathbf{s}}_n) =$ $\mathbf{F}(\tilde{\mathbf{t}}_n)\ e^{-(\tilde{\mathbf{s}}_n\cdot\tilde{\mathbf{t}}_n)}d\tilde{\mathbf{t}}_n$

$\mathbf{R}$ is analytic function on $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2,...,Re(s_n) > \gamma_n$.

Similar result for Theorem 4 can be found in [**27**]. In our result we used Lemma 3 to prove the analyticity of the n^{th} dimensional Laplace transform.

from Theorem 4 it follows that the n^{th} dimensional Laplace transform is infinitely many times differentiable and continuous on $Re(s_1) > \gamma_1, Re(s_2) > \gamma_2, ..., Re(s_n) > \gamma_n$., and hence can be differentiated of all orders inside the n^{th} dimensional Laplace integral operator.

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